Journal of Engineering Mathematics (2006) 55: 97–110
DOI 10.1007/s10665-004-3064-7 

# On the nonuniqueness of solutions to the nonlinear equations of elasticity theory

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Received 19 January 2004; accepted in revised form 6 September 2004 / Published online: 30 June 2006

**Abstract.** One-dimensional selfsimilar problems for waves in an elastic half-space generated by a sudden change of the boundary stress (the "piston" problem) and problems of disintegration of an arbitrary discontinuity are considered. For the case when small-amplitude waves are generated in a medium with small anisotropy a qualitative analysis shows that these problems have nonunique solutions when it is assumed that the solutions involve Riemann waves and evolutionary discontinuities. The above-mentioned problems are considered as limits of properly formulated problems for visco-elastic media when the viscosity tends to zero or (what is the same) that time tends to infinity. It is numerically found that all above-mentioned inviscid solutions can represent the asymptotics of visco-elastic solutions. The type of asymptotics depends on those details of the visco-elastic problem formulation which are absent when formulating inviscid selfsimilar problems. Similar considerations are made for elastic media with dispersion along with dissipation which are manifested in small-scale processes. In such media the number of available asymptotics (as $t \to \infty$) for the above-mentioned solutions depends on a relation between dispersion and dissipation and can be large. Thus, two possible causes for the nonuniqueness of solutions to the equations of elasticity theory are investigated.

**Key words:** dispersion, elasticity theory, shock wave, structure, viscosity

## 1. Introduction

Hyperbolic systems of equations expressing conservation laws play an important role in continuum mechanics. Examples of such systems are the gas-dynamic and magneto-hydrodynamic equations, as well as the equations of the nonlinear theory of elasticity. When constructing solutions of such equations one must introduce discontinuities on which the relations following from the conservation laws are satisfied. If the relations on a discontinuity do not involve any additional boundary conditions, except the conservation laws, then (as shown by P.Lax [1]) one of $n$ inequality systems relating the discontinuity velocity and characteristic velocities behind and ahead of the discontinuity should be satisfied for boundary-condition correctness. These conditions are known as the Lax conditions. In the Russian scientific literature they are referred to as evolutionary conditions. For gas dynamics these conditions were formulated by L. Landau [2].

When the Lax conditions are valid, all possible discontinuities are divided into $n$ types which are called shock waves. As shown in [1], if the relations on discontinuities and Riemann waves are represented by series expansions up to $\epsilon^3$ ($\epsilon$ characterizes a wave amplitude), then the self-similar problem on arbitrary discontinuity disintegration (the Riemann problem) has a unique solution. In addition to the Lax conditions, the condition of non-decreasing entropy should be satisfied if the entropy concept is defined for the initial system.

As seen from the above, two interesting problems arise. The first is to construct solutions for the case when it is not sufficient to consider expansions up to $\epsilon^3$, but discontinuities satisfying the Lax conditions and the condition of non-decreasing entropy are assumed to exist and used to construct solutions. The second concerns the problem of selecting discontinuities, which are considered to be implemented (admissible) for one or other type of physical conditions. The last problem is outside the framework of initial hyperbolic equations and requires the introduction of additional physical hypotheses.

The main goal of this article is to study these problems for equations of the nonlinear theory of elasticity and, not in the least, problems of nonuniqueness (obtained in Section 3) in the case of medium anisotropy in planes of wave fronts. Until now, Riemann and shock waves of small and finite amplitude have been considered for isotropic media [3–6]. One should also mention the corresponding waves in ideal magneto-hydrodynamics [7]. In this case the solution of self-similar problems for a half-space appears to be unique, and this can be explained by some degeneration of the problem due to properties of the isotropic medium in constant phase planes for waves whose solution is constructed.

Anisotropy seems to be the case in general, which is not only attributable to medium properties but also to medium predeformation. Let us note that nonuniqueness was previously obtained when solving many problems described by hyperbolic equations. For example, longitudinal wave motions of elastic media and gases with complicated equations of state [8–10] are examples in kind as well as wave motions described by first-order special equations [11]. In these papers dissipative terms with higher-order derivatives and small coefficients were included into hyperbolic equations, making it possible to consider continuous solutions (instead of discontinuities) with the parameters changing sharply in narrow zones, the so-called shock-wave structures. It was assumed that physically admissible discontinuities correspond to traveling waves having steady-state structures. This assumption makes it possible to select the unique solution of each problem.

But there are examples of partial differential equations which are constructed to demonstrate that the requirement for a structure to exist is not sufficient to obtain a unique solution [12–14]. The solutions may be influenced by dissipation or initial conditions in infinitesimally small regions. Another example of a similar nonuniqueness is the problem of wave-front propagation in combustible gas mixtures. The solution of the large-scale problem involving a combustion or detonation front depends on the mixture at ignition. The ignition process can be implemented in a narrow region whose width is of the order of the combustion or detonation front thickness.

In Section 2 the basic concepts are formulated concerning one-dimensional nonlinear small-amplitude waves in weakly anisotropic elastic media. Attention is focused on quasi-transverse waves. Equations describing wave propagation and interaction of aligned waves of this kind are obtained. The relations on a discontinuity and the condition of non-decreasing entropy are formulated. Similitude conditions are formulated which show that for a given solution in an isotropic medium one can find a similar solution with strains and stresses as small as required.

In Section 3 Riemann and shock waves are analyzed. The self-similar solution of the "piston" problem is considered when on a boundary of the elastic uniformly stressed half-space the applied stresses are suddenly changed. The solution for the Riemann and shock waves is constructed. This solution appears to be non-unique. Due to the similitude condition (Section 2), nonuniqueness in an isotropic medium exists for arbitrarily small stresses and strains. No such situations existed in previously considered problems in continuum mechanics.

In Section 4 viscous-stress terms are included in the equations of motion. Discontinuity structures are studied and it is shown that structural solutions exist for evolutionary discontinuities only. This justifies using shock waves (evolutionary discontinuities) when constructing solutions (Section 3). The results of a numerical analysis of non-self-similar solutions with self-similar non-viscous asymtotics – which are solutions obtained in Section 3 – are presented. It is shown that transition (with time) to one or other non-viscous asymptotic solution may depend on initial or boundary conditions in time or space intervals that are proportional to the viscosity coefficient. Thus, the non-viscous problem considered in Section 3 cannot settle the question which solution is implemented in the nonuniqueness case. This resembles the above-mentioned problem of combustible-gas-mixture ignition; however, the nonuniqueness in the theory of nonlinear elasticity exists (as emphasized earlier) for arbitrarily small strains and stresses.

In Section 5 problems in the nonlinear theory of elasticity are considered when the small-scale processes are determined not only by viscosity (as in Section 4), but also by dispersion with dispersion playing the greater role. Solutions of a model equation are considered which can describe nonlinear longitudinal wave propagation in a rod with a complicated strain-stress dependence. It is shown that, when dispersion effects are essential, not all discontinuities satisfying the Lax conditions are admissible, *i.e.*, have stationary structures. Instead, special discontinuities appear with an additional relation following from the requirement that the structure exists and depends on the viscosity-dispersion relation. For a given state ahead of the discontinuity, a set of such discontinuities is determined by the relation between viscosity and dispersion coefficients. By increasing the relative dispersion effect, one obtains that the number of such special discontinuities grows indefinitely. When constructing large-scale solutions of continuous waves (without viscosity and dispersion effects) and of discontinuities having structures, one obtains that the number of possible solutions corresponds to the number of special discontinuities propagating through a given medium state. Thus, there is nonuniqueness for which waves and fronts (of which solutions consist) and the number of possible solutions depend on the dissipation–dispersion relation. Such nonuniqueness of solutions did not arise in problems considered previously. Numerical analysis of non-self-similar problems for which the above-mentioned solutions can be treated asymptotically (when time tends to infinity), is presented.

## 2. Small-amplitude nonlinear waves

Let us restrict ourself to the case of nonlinear small-amplitude waves in elastic media with small anisotropy. The anisotropy can be due to medium properties or previous deformation. Thus, the presence of anisotropy can be considered in generality. It is known that linear waves in isotropic media are divided into longitudinal and transverse. Transverse waves are waves of two types (of various polarizations) propagating at the same velocity. For small anisotropy, waves are divided into quasi-longitudinal and quasi-transverse, but, in the case under consideration, quasi-transverse waves have two slightly different velocities. The nonuniqueness is related to the behavior of quasi-transverse waves and, in what follows, only such waves will be considered.

As in the case of the Hopf equation, describing the waves of one characteristic family, one can obtain the equations which describe the waves of two characteristic families of quasi-transverse waves when it is assumed that the disturbances carried on the remaining characteristics are sufficiently small. In the general case the equations taking into account the main order of a small nonlinearity and anisotropy and describing quasi-transverse waves in uniform elastic media are [15, Chapter 7], [16].

$$\frac{\partial u_\alpha}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_\alpha}\right)=0, \quad \alpha=1,2, \tag{1}$$

$$H=\frac{1}{2}f(u_1^2+u_2^2)-\frac{1}{4}\varkappa(u_1^2+u_2^2)^2+\frac{1}{2}g(u_2^2-u_1^2). \tag{2}$$

Here $u_\alpha=\partial w_\alpha/\partial x$, $x$ is the Lagrangian coordinate normal to wave fronts, $w_\alpha$ are the tangential components of the displacement vector, $f$, $\varkappa$, $g$ are combinations of coefficients in the internal-energy expansion, $f$ is the characteristic velocity in the case when the nonlinearity and anisotropy are absent, $\varkappa$ is the nonlinearity coefficient independent of anisotropy (since nonlinearity and anisotropy are small), $g$ is the coefficient of anisotropy assumed to be small. Since anisotropy and $u_i$ are small, only the second-order terms responsible for anisotropy are taken into account in expression (2). There are such terms for almost all types of anisotropy if the chosen $x$-axis direction does not coincide with any of the special directions of the medium. Besides, such second-order terms can be generated by small predeformation in planes normal to the $x$-axis. These terms can be reduced to the form of the last term in expression (2) when we turn the $(x, y)$-axes and single out the isotropic part. By choosing the labelling of $u_\alpha$, one can always get: sign $g=$ sign $\varkappa$. If $u_i$ is of the order of $\varepsilon$, then $g$ should be of the order of $\varepsilon^2$ for terms responsible for nonlinearity and anisotropy to be of the same order. For originally isotropic media the coefficient $g$ is proportional (with proportionality factor of the order of unity) to $\varepsilon_1-\varepsilon_2$ where $\varepsilon_1$ and $\varepsilon_2$ are the principal strains in the wave plane. The remaining quantities changing in the wave can be expressed in terms of $u_1$ and $u_2$. In particular, for longitudinal deformation the following equality holds

$$u_3=h(u_1^2+u_2^2)+\text{const}, \quad u_3=\frac{\partial w_3}{\partial x} \tag{3}$$

where $w_3$ is the displacement in the $x$-direction, the coefficient $h$ is determined by medium elastic properties. Equations (1) form a nonlinear hyperbolic system, at least, for sufficiently small $u_i$. The corresponding discontinuity conditions follow from the momentum conservation laws

$$\left[\frac{\partial H}{\partial u_\alpha}\right]=W[u_\alpha], \quad \alpha=1,2, \tag{4}$$

$$[u_3]=h[u_1^2+u_2^2]. \tag{5}$$

Here $W$ is the velocity of the discontinuity.

Equations (3) and (5) make it possible to eliminate $u_3$. As usual, square brackets denote jumps of parameters: $[a]=a^+-a^-$. Equations (4) express conservation laws for momentum components tangent to the discontinuity. An entropy change in a discontinuity is determined from the energy conservation law. The condition of nondecreasing entropy is of the form

$$\left\{\left(\frac{\partial H}{\partial u_\alpha}\right)^+ +\left(\frac{\partial H}{\partial u_\alpha}\right)^-\right\}[u_\alpha]\leq 0. \tag{6}$$

By a Galileian transformation and by choosing for $t$, $x$ and $u_\alpha$ the scales of $T$, $L$ with $L/T=|g|$ and $\sqrt{g/\varkappa}$, respectively, one can obtain the canonical form of the system equations (1–3) with $f=0$, $g/\varkappa=1$, $\varkappa=\pm 1$. As follows from the above, if we have two Cauchy problems in two media with initial data $u_i^{0(1,2)}(x)$, such that $u_\alpha^{0(1)}(x/L^{(1)})\Big/\sqrt{g^{(1)}/\varkappa^{(1)}}=u_\alpha^{0(2)}\left(x/L^{(2)}\right)\Big/\sqrt{g^{(2)}/\varkappa^{(2)}}$ (with arbitrary $L^{(1)}$ and $L^{(2)}$), then the solutions are similar [17, Section 7.4.4]. Since in isotropic media $g$ is of the order of $\varepsilon_1-\varepsilon_2$ and can be arbitrarily small, all the following considerations and, in particular, the conclusion on nonuniqueness, are also valid for problems in isotropic media with arbitrarily small strains.

For a self-similar piston the problem is posed as follows:

$$u_\alpha = U_\alpha = \text{const} \quad \text{at} \quad t=0 \quad \text{for} \quad x>0,$$
$$u_\alpha = u_\alpha^* = \text{const} \quad \text{at} \quad x=0 \quad \text{for} \quad t>0.$$

The solution depends on $x/t$ and should involve both Riemann waves and discontinuities.

The problem of disintegration of an arbitrary discontinuity is reduced to two piston problems with initially unknown data for $x=0$, which should be obtained by boundary conditions at this point. If nonuniqueness takes place for the piston problem, it also takes place for the arbitrary discontinuity-disintegration problem.

## 3. Riemann and shock waves. Solution of piston problem

Riemann waves are solutions to a hyperbolic system (in our case of Equations (1)) of the form

$$u_i = u_i(\theta(x,t)) \tag{7}$$

with an unknown function $\theta(x,t)$. Riemann waves in elastic media were considered in [18, 3, 5, Chapter III]. In the case of small-amplitude waves in weakly anisotropic media it follows [15, 17, 19, Chapter 3; Section 7.4.5] from (1), (7) that

$$(H_{\alpha\beta} - c\delta_{\alpha\beta})\frac{\mathrm{d}u_\beta}{\mathrm{d}\theta} = 0, \quad c = -\frac{\partial\theta/\partial t}{\partial\theta/\partial x}, \quad H_{\alpha\beta} = \frac{\partial^2 H}{\partial u_\alpha \partial u_\beta} \qquad \alpha,\ \beta = 1, 2. \tag{8}$$

For a solution to be nontrivial $c$ should be one of the eigenvalues of the matrix $H_{\alpha\beta}$ and $\mathrm{d}u_\beta/\mathrm{d}\theta$ should be components of the corresponding eigenvector. It follows from the second equation of (8) that

$$\frac{\partial\theta}{\partial t} + c(u_1(\theta), u_2(\theta))\frac{\partial\theta}{\partial x} = 0. \tag{9}$$

The characteristics of the above equation are straight lines in the $(x,t)$-plane on which $\theta$ and $u_1, u_2$ are constant. For a selfsimilar solution we have $\theta = x/t$, $c = x/t$, so that $c$ decreases with $t$ for $x$ fixed.

In the $(u_1, u_2)$-plane integral curves of the first of equations (8) are tangent to eigenvectors of the symmetric matrix $\|H_{\alpha\beta}\|$ at each point and form two families of mutually orthogonal curves. The axes $u_1$ and $u_2$ are symmetry axes of integral curves and $c(u_1,u_2) = c(u_1,-u_2) = c(-u_1,u_2) = c(-u_1,-u_2)$. At $u_1 = 0$, $u_2 = \pm\sqrt{G}$, $G = g/\varkappa$ the eigenvalues coincide: $c_1 = c_2$. These points are singular points of the equations. Integral curves are given in Figure 1.

For $\varkappa > 0$ elliptic-like curves correspond to fast waves $c = c_2(u_1,u_2)$ and the hyperbolic-like curves correspond to slow waves $c = c_1(u_1,u_2)$, $c_1 \le c_2$. The arrows indicate the direction of decreasing $c$ along the corresponding integral curves.

The discontinuities in $u_1$ and $u_2$ are described by Equation (4). If $u_1 = U_1, u_2 = U_2$ ahead of the discontinuity, then behind the discontinuity $u_1$ and $u_2$ belong to a curve on the $(u_1,u_2)$-plane which can be denoted as the shock adiabat [15, Chapter 4], [17, Section 7.4.6], [20, 21]

$$(u_1^2 + u_2^2 - R^2)(U_1u_2 - U_2u_1) + 2G(u_1 - U_1)(u_2 - U_2) = 0, \qquad R^2 = U_1^2 + U_2^2. \tag{10}$$

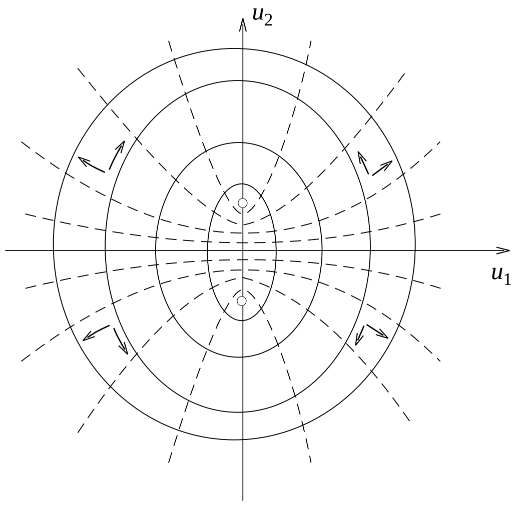


*Figure 1.* Integral curves of quasitransverse Riemann waves.

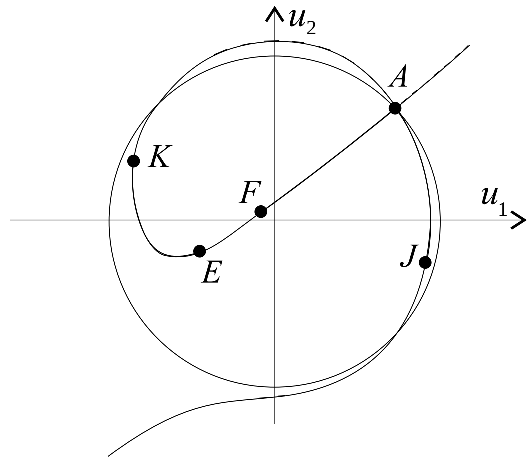


*Figure 2.* Hugoniot curve of quasi-transverse shocks.

Inequality (6) of nondecreasing entropy gives

$$\varkappa(u_1^2 + u_2^2 - R^2)\{(u_1 - U_1)^2 + (u_2 - U_2)^2\} \leq 0. \tag{11}$$

The typical form of the shock adiabat is shown in Figure 2. If an initial point $(U_1, U_2)$ is reflected from any of the $(u_1, u_2)$-axes (*i.e.*, $U_1$ or $U_2$ changes sign), then the shock adiabatic and $W$ on it are also reflected from the same axis.

In the absence of external effects, relations (4) should be satisfied at all discontinuities. If there are no additional conditions at the discontinuity, the correctness conditions in a vicinity of the discontinuity for the linearized problem give restrictions on the velocity $W$:

$$\begin{aligned} &(a) \quad c_2^- \leq W, \qquad c_1^+ \leq W \leq c_2^+, \\ &(b) \quad c_1^- \leq W \leq c_2^-, \qquad W \leq c_1^+. \end{aligned} \tag{12}$$

Inequalities (12) are called the Lax conditions [1] or (*a priori*) evolutionary conditions. The first group of inequalities, (a), determines fast shocks and the second (b), determines slow shocks. For the case under consideration, the entropy condition (11) follows from the evolutionary conditions (12).

Inequalities (12) can be represented in a diagram (Figure 3) in which both axes correspond to shock velocities $W$. Discontinuities which do not satisfy inequalities (12) can not exist due to non-correctness which manifest itself in unlimitedly fast growth of disturbances. The parameters $c_1^-$, $c_2^-$ and $W$ can be shown on the horizontal axis in the same fixed scale, but in the vertical direction the scales for velocities do not remain constant and only inequalities between $c_1^+$, $c_2^+$ and $W$ are retained. The hatched rectangles correspond to inequalities (12). The shock adiabatic is qualitatively shown on the diagram for $\varkappa > 0$ for the case of sufficiently large $U_1/\sqrt{G}$ and $U_2/\sqrt{G}$. The points having the same designations in Figures 2 and 3 correspond to each other. The evolutionary segments of the shock adiabatic are shown as bold curves.

Solutions of a piston problem were analyzed in [15, Chapter 5], [17, Section 7.4.7], [22, 23]. For fixed initial values $U_1, U_2$, the whole $u_1, u_2$ plane is divided into domains in such a way

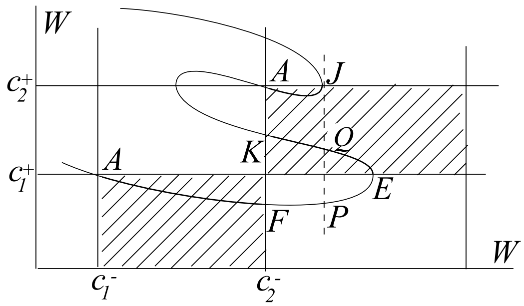


*Figure 3.* Evolutionary diagram.

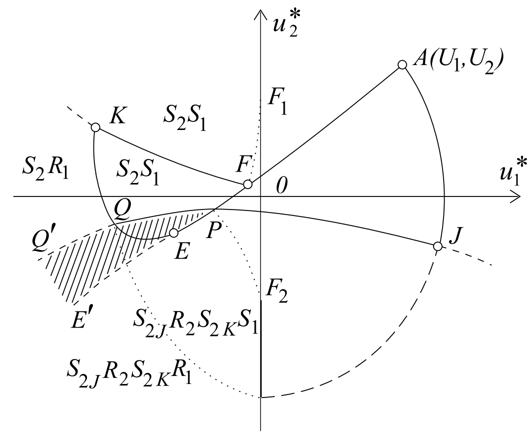


*Figure 4.* Nonuniqueness domain.

that, if boundary values $u_1^*, u_2^*$ belong to one of them, then a definite combination of Riemann waves and shocks gives the solution. If $U_1/\sqrt{G}$ and $U_2/\sqrt{G}$ are high (it is sufficient to have $U_1/\sqrt{G} > 2$, $U_2/\sqrt{G} > 2$), there exists a domain which is the overlapping of two domains with different solutions. In Figure 4, where domains for $\varkappa > 0$ are shown, the domain with two solutions is hatched.

In Figure 4 the evolutionary parts of the shock adiabatic with an initial point $A(U_1, U_2)$ are shown by bold curves. The wave combinations in domains neighboring the nonuniqueness domain are symbolically denoted as $S_2S_1$ and $S_{2J}R_2S_{2K}S_1$. A fast shock is denoted by $S_2$ (in the case under consideration it corresponds to a jump from $A$ to some point of the interval $KE$), $S_1$ denotes a slow shock, $S_{2J}$ is a fast shock $A \to J$, $R_2$ is a selfsimilar fast Riemann wave, $S_{2K}$ is a shock of the same type as $A \to K$. The solutions from neighboring domains extend into the hatched domain. They differ everywhere, except the line $PQQ'$ where the solutions coincide.

Let us note that one can conclude that solutions are nonunique in view of Figure 3. In [24] the sufficient condition for nonunique solutions to the conservation law system to exist was obtained. Is the case under consideration this condition is satisfied since only one evolutionary interval $QE$ (Figure 3) is above the nonevolutionary interval $PE$ of the shock adiabatic.

One can note that the more simple solution $S_2S_1$ in the nonuniqueness domain includes a jump $S_1$ from the initial point $A$ to some point of interval $QE$. The interval is located in the domain where the solution $S_{2J}R_2S_{2K}S_1$ also holds. Thus, in the framework of the considered "ideal" (inviscid, nondispersive) nonlinear theory of elasticity the fast shock $S_2$ can disintegrate. However, whether this disintegration takes place when the nonideal nature of elastic media is taken into account remains a question for the next sections.

The main results of Sections 2 and 3 were previously obtained using a system of nonlinear elasticity equations. For simplicity, the results are represented here using the simplified Equations (1), (4).

## 4. Waves in viscous-elastic media. Inviscid limit

It is shown in this section that solutions of the piston problem constructed in Section 2 are asymptotics of visco-elastic solutions for the viscosity tending to zero or (which is the same) for fixed viscosity and $t \to \infty$. There are many models of visco-elastic media, including those that are described by integro-differential equations. In what follows the Kelvin–Voigt model will be used. This model leads to the simplest equations which are convenient for the initial analysis. At the same time, one can believe that results obtained for the Kelvin–Voigt model appear to be valid for some other models as well.

Consider small-amplitude one-dimensional quasi-transverse waves propagating in the positive $x$-direction; they are described by the following equations [15, Chapter 8], [17, Section 7.4.8]

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_i}\right) = \mu \frac{\partial^2 u_i}{\partial x^2}, \quad i = 1, 2. \tag{13}$$

Here viscous terms with constant viscosity coefficient $\mu$ are added to Equations (1). The viscosity is taken to be isotropic since anisotropy is assumed to be small and all the effects in Equations (13) are taken into account by the main terms only. Dissipative processes due to thermo-diffusion are not taken into account since temperature variations in quasi-transverse waves are only of second order in a wave amplitude.

Equations (13), like (1), can be transformed to canonical form in which the equality $\mu = 1$ is valid, in addition to equalities $f = 0$, $\varkappa/g = 1$, $\varkappa = \pm 1$. When Equations (1) are transformed, the scales $T$ and $L$ are related by $L/T = g$ only. In the case under consideration the relation $L^2/T = \mu$ should be added. Thus, $L = \mu/g$, $T = \mu/g^2$.

The problem of shock-wave structure was considered in the framework of Equations (13) [25,26]. It was found that stationary structures of the form $u_i = u_i(x - Wt)$ correspond to all evolutionary shocks and only to these.

The generalized piston problem

$$\begin{aligned}
&t = 0, \ \ x > 0 && u_i = U_i = \text{const}, \\
&0 \le t \le \delta, \ \ x = 0 && u_i = u_i^0(t), \ \ u_i^0(0) = U_i, \quad u_i^0(\delta) = u_i^*, \\
&t > \delta, \ \ x = 0 && u_i = u_i^* = \text{const}
\end{aligned} \tag{14}$$

is considered. The value of $\delta$ can be arbitrarily chosen, but it is evident that it can be chosen proportional to $\mu$ and is restricted by time scales of viscous and nonlinear processes which occur until a selfsimilar solution is achieved. As $t \to \infty$ (or $\mu \to 0$ for fixed $t$), solutions of generalized piston problems tend to selfsimilar solutions of problems in the ideal theory of elasticity considered in Section 3.

Solutions of generalized piston problems for various functions $u_i^0(t)$ were numerically constructed [26–28]. When two selfsimilar solutions exist for $U_i$, $u_i^*$, it was found that in most cases the simple selfsimilar solution $S_1S_2$ arises asymptotically as $t \to \infty$. Selfsimilar solutions of the other type emerge if the trajectory of the point $(u_1^0(t), u_2^0(t))$ goes into the nonuniqueness domain from below, where only the selfsimilar solution $S_J R_1 S_K S_2$ (or $R_2$) exists. Moreover, for this complicated asymptotic to arise, the point $(u_1^0(t), u_2^0(t)$ should stay in the domain below for a sufficiently long time.

The problem of two-shock-structure interaction has been numerically analyzed [29]. In the case of nonuniqueness more simple asymptotics emerge.

A stability problem for a shock which can disintegrate was considered. It was shown that such shocks are stable with respect to infinitesimal disturbances [30]. In visco-elastic media, interactions of waves representing structures of such shocks with finite amplitude one-dimensional and non-one-dimensional disturbances were numerically considered in [31,32]. As was

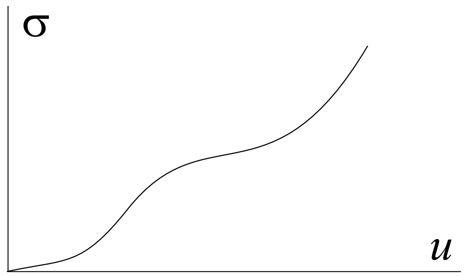


*Figure 5.* Tension-strain dependence.

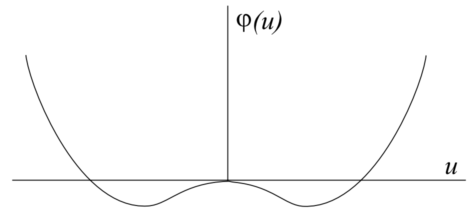


*Figure 6.* Graph of $\varphi(u)$ function.

shown in a number of numerical experiments, for such a wave to disintegrate, one-dimensional disturbances of amplitude comparable with a shock amplitude are required. In non-one-dimensional cases it is shown that compactly supported disturbances can destroy the waves in some domains for some time, but later the waves return to their initial forms [17].

These results confirm that in the case of nonuniqueness each of solutions to nonlinear elastic problems can be implemented, at least, when they are considered as inviscid limits of problems with viscosity.

## 5. Long waves in nonlinear media with dispersion and dissipation effects

In this section the simultaneous influence of dispersion and dissipation on the formation of large-scale selfsimilar nonlinear solutions of the elasticity theory is analyzed. The terms responsible for dispersion and dissipation in the equations lead to the appearance of some typical linear scale $l$ in the solutions. When large-scale phenomena are considered with space scale $L >> l$, then the processes concerned with dissipation and dispersion can be manifested only in the cases when the solutions are essentially changing in narrow (as compared to $L$) regions. From the large-scale point of view such changes of the solutions should be considered as discontinuities and continuous solutions in a vicinity of this discontinuity represent its structures. If solutions in the large-scale approximation are constructed for continuous waves and discontinuities having structures, then it appears [33–35] that such solutions are nonunique. The greater the role of dispersion (as compared to dissipation), the greater the number of possible solutions will be. Solving the non-selfsimilar problems numerically (with dispersion and dissipation taken into account) makes it possible to find solutions implemented under specific conditions.

Consider first the propagation of longitudinal waves in visco-elastic rods with complicated elastic properties. Let $\sigma$ be the total tension in a rod section and $u$ be a longitudinal strain. Assume that the graph of $\sigma(u)$ has two inflection points (Figure 5).

The strain $u$ is $\partial w/\partial x$ where $w$ is the displacement along the rod axis (the $x$-axis). For long waves $u$ depends on $x$ and $t$. The function $\sigma(u)$ is assumed to be a near linear one. The equation describing the wave propagation can be written in the form [33]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 \sigma(u)}{\partial x^2} + \beta^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + \alpha \frac{\partial^3 u}{\partial t \partial x^2}. \tag{15}$$

Here $x$ is the Lagrangian coordinate, the term with $\beta^2$ represents main dispersion effects for long waves and the last term represents viscosity effects according to the Kelvin–Voigt model. For a circular rod Rayleigh obtained that the relation $\beta = \nu r$ holds, where $\nu$ is the Poisson coefficient and $r$ is the rod radius. The coefficients $\beta$ and $\alpha$ for small-amplitude waves can be taken to be constant.

For small nonlinearity, independent equations can be derived for waves propagating in the positive and negative directions of the $x$-axis (as when deriving the Burgers and Korteweg-de Vries equations). For waves propagating in the positive $x$-direction we have

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u)}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2}-m\frac{\partial^3 u}{\partial x^3}, \tag{16}$$
$$\mu=\frac{1}{2}\alpha,\quad m=\frac{1}{2}c_0\beta^2,\quad \varphi(u)=\int c(u)\,\mathrm{d}u\quad c(u)=\sqrt{\frac{\mathrm{d}\sigma}{\mathrm{d}u}}.$$

Here $c_0$ is a mean value of $c(u)$ in the interval of $u$-variations. Since the right-hand side of Equation (16) is small, some approximations were made. In particular, $\partial/\partial t$ was replaced by $-c_0\partial/\partial x$.

The graph of the function $\varphi(u)$ has the same two inflection points as the graph of $\sigma(u)$. Indeed, $\varphi''=\sigma''/\sqrt{2\varphi'}$. After a Galileian transformation the function $c(u)$ obtains an additional constant and the function $\varphi(u)$ obtains a linear term. Thus, the graph of $\varphi(u)$ can be represented, as shown in Figure 6.

In calculations the following form is used

$$\varphi(u)=u^4-u^2. \tag{17}$$

For very long waves the right-hand side of Equation (16) can be neglected and the corresponding discontinuity condition takes the form

$$\frac{\partial u}{\partial t}+c(u)\frac{\partial u}{\partial x}=0,\quad W=\frac{[\varphi(u)]}{[u]},\quad c(u)=\frac{\mathrm{d}\varphi}{\mathrm{d}u}. \tag{18}$$

Here $c(u)$ is the characteristic velocity and $W$ is the shock velocity.

Stationary structures of discontinuities are described as follows

$$u=u(\xi),\quad \xi=x-Wt,$$
$$m\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}-\mu\frac{\mathrm{d}u}{\mathrm{d}\xi}=F(u),\quad F(u)=W(u-u^-)-(\varphi(u)-\varphi(u^-)), \tag{19}$$
$$\lim_{\xi\to\infty}u(\xi)=u^-,\quad \lim_{\xi\to-\infty}u(\xi)=u^+.$$

The set of values behind the shock $u^+$ for which the solution of the structure problem (19) exists for a given value ahead of the shock $u^-$ was investigated in [33,36]. It consist of intervals and separate points. The set of $u^+$ is shown in Figure 7 as bold intervals and points on the graph $\varphi(u)$ for $u^-=-1$, $m=1\cdot 3$, $\mu=0\cdot 05$. The number and positions of intervals and points depend on $u^-$ and $m\mu^{-2}$. Both the number of intervals and the number of points tend to infinity as $m\mu^{-2}\to\infty$. If $0<u^-<u^+$ and $F(u)>0$ on $(u^-,u^+)$, there exists a structure solution of the discontinuity $u^-\to u^+$.

Differential equation (18) has continuous solutions as well, which have no tendency to discontinuity formation. For these solutions the characteristic velocity $c(u(x))$ is a nondecreasing function of $x$.

When solutions to "ideal" systems like (18) are constructed, the hypothesis is frequently made that only discontinuities with stationary structures can exist. In many cases the hypothesis makes it possible to select a unique solution of the problem [8,11]. In particular, this is the case for solutions to Equations (18) when the discontinuity structure is described by (19) with $m=0$ [11].

For sufficiently large values of $m\mu^{-2}$ the requirement of the structure to exist does not determine the solution to Equations (18) uniquely. Indeed, consider a problem of the disintegration of an arbitrary discontinuity: at $t=0$, $u=u^-=-1$ for $x>0$ and $u=u^+$ for $x<0$. In the graph of the function $\varphi(u)$ the first state is denoted by $A$ and the second by $F$. In the situation corresponding to Figure 7 there are five piece-wise constant solutions. One solution involves only one discontinuity $A\to F$. Four other solutions involve two discontinuities $A\to C_i$ and $C_i\to F$, $i=1,2,3,4$. As mentioned, the number of separate points $C_i$ increases with $m\mu^{-2}$. The number of solutions with shocks to which stationary structures correspond also grows.

Similar problems arise in a number of other physical problems with the dispersion in discontinuity structures. The propagation of quasi-transverse nonlinear elastic waves in composites [34] and nonlinear electromagnetic waves in magnets [35,37] are such problems.

A number of numerical solutions to Equation (16) was analyzed to determine what kind of arbitrary discontinuity decomposition takes place and under which circumstances. The solutions to the Cauchy problem for Equation (16) of the form

$$\begin{aligned} &u=u^- \quad \text{for} \quad x>a, \qquad u=u^+ \quad \text{for} \quad x<a,\\ &u=u_0(x/a) \quad \text{for} \quad -1\le x/a\le 1,\ \ u_0(-1)=u^+,\ \ u_0(1)=u^- \end{aligned} \tag{20}$$

were numerically obtained for various functions $u_0(x/a)$. Obviously, for the fixed function $u_0$ the asymptotic to the solution is not changed for $a$ greater than $a^*$ which is determined by the space scale of the asymptotic shock structures. The Cauchy problem (20) can be called the generalized problem of an arbitrary discontinuity decomposition. The selfsimilar asymptotics as $t\to\infty$ were obtained.

As results of numerical experiments it has been found that:

(a) Stationary structures of discontinuities are stable. If the initial function $u_0(x/a)$ is a slightly perturbed stationary structure, the perturbation tends to zero with time growing. The stability of stationary structures means that, for every solution involving discontinuities with stationary structures, there exists a domain of such functions $u_0(x/a)$ in a functional space for which solutions of the Cauchy problem (16), (20) tend to the abovementioned selfsimilar solutions.

(b) Along with discontinuities with stationary structures, there are discontinuities with unsteady structures that are periodic in time. Since the perturbations do not propagate from the discontinuity structure to infinity, the second equation of (18) remains valid, but in this case $W$ is a mean velocity. When $u^-=-1$, an oscillating structure exists if $u^+>u(P)$ ($P$ is a point in Figure 7).

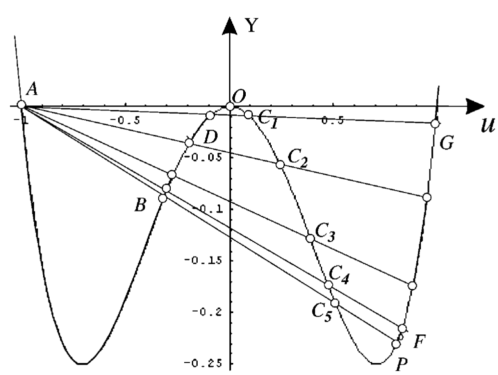


*Figure 7.* Set of values $u$ behind shocks with stationary structure on the shock adiabat.

(c) If for given $u^-$, $u^+$ a selfsimilar asymptotic involving discontinuities with unsteady structure exists, this asymptotic arises, as a rule, as $t \to \infty$, for a generalized problem of arbitrary discontinuity decomposition with nonspecial initial functions $u_0(x)$. One can say that, in a functional space of functions $u_0(x)$, the attraction domain of selfsimilar asymptotics involving discontinuities with unsteady structures is greater than the attraction domain of selfsimilar asymptotics involving discontinuities with stationary structures.

Let us note that analytical and numerical investigations of the generalized Cauchy problem on an arbitrary discontinuity decomposition show that for $m\mu^{-2} \gg 1$ there are many different asymptotic cases depending on $u_0(x)$.

## 6. Conclusions

The nonuniqueness of selfsimilar solutions of the equations of nonlinear elasticity has been demonstrated. It can occur for small perturbations of a homogeneous state. The analysis shows that there are two different reasons for the nonuniqueness to exist. For quasi-transverse small-amplitude waves in a weakly anisotropic medium, solutions of selfsimilar problems are constructed when discontinuities used in solutions satisfy the Lax conditions. It is shown that such discontinuities (and only such) have structures due to the viscosity. It appears that under these conditions there are two different solutions for some range of parameters determining the problem. It has been shown numerically that non-self-similar solutions of viscous-elastic equations which tend to inviscid asymptotics can be different depending on the details of the initial and boundary conditions. The details are concerned with time and space intervals which depend on the viscosity and tend to zero with viscosity. The details determining the type of the viscous asymptotic are not included in the hyperbolic problem formulation. Therefore one should use more detailed equations, the viscous-elastic equations in the case under consideration.

The dispersion effect for longitudinal waves in a rod leads to discontinuities which do not satisfy the Lax conditions. The number of such discontinuities of different type is determined by the relative influence of two competing effects in a shock-wave structure – dispersion and viscosity – and this number can be large. Besides, the numerical analysis of non-self-similar problems with viscosity and dispersion shows that there are discontinuity structures with internal time-depending oscillations in addition to the above-mentioned discontinuities with steady-state (stationary) structures. This leads to multiple nonuniqueness of the problem. As before, to determine a correct self-similar asymptote, one should not restrict oneself to studying the solutions of hyperbolic systems but also consider more complicated equations. In contrast to the previous case, the non-Laxian discontinuities are not known beforehand now and should be found from the requirement that a structure should exist.

From the problems considered above one can conclude that nonuniqueness of solutions to hyperbolic systems of continuum mechanics, including nonlinear elasticity equations, is the rule rather than the exception. In cases of nonuniqueness one should consider more complicated systems of equations which have continuous solutions and take into account real small-scale processes. Such detailed equations and initial data are required to predict the behavior of global solutions.

## Acknowledgements

This research is supported partly by the Russian Foundation for Basic Research(02-01-00729 and 02-01-00613) and by the programs "Nonlinear dynamics" of Russian Academy of Sciences and "Leading Scientific Schools" of Russian Federation (the grant NSh-1697.2003.1).

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